

COMPLEX SYMMETRIC COMPOSITION OPERATORS WITH AUTOMORPHIC SYMBOLS

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ABSTRACT. In this paper we show that a composition operator C_φ cannot be complex symmetric on Hardy-Hilbert space $H^2(D)$ when φ is an elliptic automorphism of order 3 and not a rotation. This completes the project of finding all invertible composition operators which are complex symmetric $H^2(D)$.

1. INTRODUCTION

Let T be a bounded operator on a complex Hilbert space \mathcal{H} . Then T is called complex symmetric if there exists a conjugation C such that $T = CT^*C$. Here a conjugation is a conjugate-linear, isometric involution on \mathcal{H} . The operator T may also be called C -symmetric if T is complex symmetric with respect to a specific conjugation C . For more details about complex symmetric operators one may turn to [4] and [5].

In this paper, we are particularly interested in the complex symmetry of composition operators induced by analytic self-maps of D . This subject was started by Garcia and Hammond in [3].

Recall that for each analytic self-map φ of the unit disk D , the composition operator given by

$$C_\varphi f = f \circ \varphi$$

is always bounded on $H^2(D)$. Here, the Hardy-Hilbert space $H^2(D)$ is the set of analytic functions on D such that

$$\|f\|_{H^2}^2 = \sup_{0 < r < 1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty.$$

Several simple examples of complex symmetric composition operators on $H^2(D)$ arise immediately. For example, every normal operator is complex symmetric (see [4]), so when $\varphi(z) = sz$ with $|s| \leq 1$, C_φ is normal hence complex symmetric on $H^2(D)$. Also, Theorem 2 in [6] states that each operator satisfying a polynomial equation of order 2 is complex symmetric. So when φ is an elliptic automorphism of order 2, then $C_\varphi^2 = I$, thus C_φ is complex symmetric on $H^2(D)$. In [7] Waleed Noor find the conjugation C such that C_φ is C -symmetric when φ is an elliptic automorphism of order 2.

2010 *Mathematics Subject Classification.* primary 47B33, 46E20; Secondary 47B38, 46L40, 32H02.

Key words and phrases. Composition operator; Complex symmetry; Elliptic automorphism.

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This work was supported in part by the National Natural Science Foundation of China (Grant Nos. 11371276; 11301373; 11401426).

However, we are still far away from our final destination: finding out all composition operators which are complex symmetric on $H^2(D)$. The first step is, of course, to determine the complex symmetric composition operators induced by automorphisms. Bourdon and Waleed Noor considered this problem in [2]. The next Proposition is Proposition 2.1 in [2]

Proposition 1.1. *Let φ be a self-map of D . If C_φ is complex symmetric on $H^2(D)$, then φ has a fixed point in D .*

Particularly, if φ is either a parabolic or a hyperbolic automorphism, then C_φ cannot be complex symmetric on $H^2(D)$.

Thanks to this proposition, we only need to investigate the elliptic automorphisms of the unit disk D . It turns out that things depend much on the orders of the automorphisms. The next proposition is one of the main result in [2].

Proposition 1.2. *Let φ be an elliptic automorphism of order q . If $q = 2$, then C_φ is always complex symmetric on $H^2(D)$. If $4 \leq q \leq \infty$, then C_φ is complex symmetric on $H^2(D)$ only if φ is a rotation.*

However, the order 3 elliptic case remains as an open question, which is posed by Bourdon and Waleed Noor in [2]:

Question 1.3. *Is C_φ complex symmetric on $H^2(D)$ when φ is an elliptic automorphism of order 3?*

The aim of this paper is to solve this problem. By doing this we complete the project of finding out all invertible composition operators which are complex symmetric $H^2(D)$.

In our main result Theorem 3.5, we prove that if φ is an elliptic automorphism of order 3 and not a rotation, then the composition operator C_φ cannot be complex symmetric on $H^2(D)$. Then we can come to our final result as follows. It will be given as Corollary 3.6 in the third section.

Main Result. *Suppose φ is an automorphism of D . Then C_φ is complex symmetric on $H^2(D)$ if and only if φ is either a rotation or an elliptic automorphism of order two.*

2. PRELIMINARY

The Hardy space $H^2(D)$ is naturally a Hilbert space, with the inner product

$$\langle f, g \rangle = \sup_{0 < r < 1} \int_0^{2\pi} f(re^{i\theta}) \overline{g(re^{i\theta})} \frac{d\theta}{2\pi}.$$

For each $w \in D$, let

$$K_w(z) = \frac{1}{1 - \overline{w}z}.$$

Then $K_w \in H^2(D)$ is the reproducing kernel at the point w , i.e.,

$$\langle f, K_w \rangle = f(w)$$

for all $f \in H^2(D)$.

It is well known that the automorphisms of the unit disk D fall into three categories: parabolic and hyperbolic automorphisms have not fixed point in D , and besides them are the elliptic automorphisms who have a unique fixed point in D .

Definition 2.1. The order of an elliptic automorphism φ is the smallest positive integer such that $\varphi_n(z) = z$ for all $z \in D$. Here $\varphi_n = \varphi \circ \varphi \circ \dots \circ \varphi$ denotes the n -th iterate of φ . If no such positive integer exists, then φ is said to have order ∞ .

Note that if the order of an automorphism φ is one, then φ is identity on D . If φ has order two, then φ is of the form

$$\varphi(z) = \varphi_a(z) = \frac{a - z}{1 - \bar{a}z}$$

for some $a \in D$.

Remark 2.2. The notation φ_a will be used throughout this paper. φ_a is the involution automorphism exchanges 0 and a .

By Propositions 1.1 and 1.2, we only need to be concerned with the elliptic automorphisms that have order 3. Moreover, if the fixed point of a automorphism is 0, then it is a rotation and of course complex symmetric. So throughout this paper we will always assume that φ is an elliptic automorphism of order 3 with fixed point $a \in D \setminus \{0\}$.

For a complex symmetric operator T , one should keep the following simple result in mind.

Lemma 2.3. *Suppose T is C -symmetric on \mathcal{H} . Then $\lambda \in \mathbb{C}$ is a eigenvalue of T if and only if $\bar{\lambda}$ is a eigenvalue of T^* . Moreover, the conjugation C maps the eigenvectors subspace $\text{Ker}(T - \lambda)$ onto $\text{Ker}(T^* - \bar{\lambda})$.*

Proof. One only need to note that $T = CT^*C$ implies $T - \lambda = C(T^* - \bar{\lambda})C$. \square

The next lemma gives the eigenvectors of C_φ on $H^2(D)$ when φ is an elliptic automorphism of order 3. The main calculation of this lemma is done in [2], with the help of Theorem 9.2 in Cowen and MacCluer's book [1].

Lemma 2.4. *Suppose φ is an elliptic automorphism of order 3 with fixed point $a \in D$. Let $\Lambda_m = \text{Ker}(C_\varphi - \varphi'(a)^m)$ and $\Lambda_m^* = \text{Ker}(C_\varphi^* - \overline{\varphi'(a)^m})$ for $m = 0, 1, 2$, then*

$$\Lambda_m = \overline{\text{span}}\{\varphi_a^{3j+m}; j = 0, 1, 2, \dots\}$$

and

$$\Lambda_m^* = \overline{\text{span}}\{e_{3j+m} - ae_{3j+m-1}; j = 0, 1, 2, \dots\},$$

where $e_{-1} = 0$ and $e_k = K_a \varphi_a^k$ for $k = 0, 1, 2, \dots$

Proof. Let $\tau = \varphi_a \circ \varphi \circ \varphi_a$. Then τ is a rotation of order 3, that is, $\tau(z) = \varphi'(a)z$. So $C_\tau z^k = \varphi'(a)^k z^k$ for $k = 0, 1, 2, \dots$. Moreover, since $C_\tau^* = C_{\tau^{-1}}$ where $\tau^{-1}(z) = \overline{\varphi'(a)}z$, we also have $C_\tau^* z^k = \overline{\varphi'(a)^k} z^k$. Thus

$$z^k \in \text{Ker}(C_\tau - \varphi'(a)^k) \cap \text{Ker}(C_\tau^* - \overline{\varphi'(a)^k})$$

for $k = 0, 1, 2, \dots$

Now by the definition of τ we have $C_\varphi C_{\varphi_a} = C_\tau C_{\varphi_a}$ and $C_\varphi^* C_{\varphi_a}^* = C_\tau^* C_{\varphi_a}^*$, so $C_{\varphi_a} z^k \in \text{Ker}(C_\varphi - \varphi'(a)^k)$ and $C_{\varphi_a}^* z^k \in \text{Ker}(C_\varphi^* - \overline{\varphi'(a)^k})$. Since $\varphi'(a)^3 = 1$ one can get

$$\overline{\text{span}}\{C_{\varphi_a} z^{3j+m}; j = 0, 1, 2, \dots\} \subset \Lambda_m$$

and

$$\overline{\text{span}}\{C_{\varphi_a}^* z^{3j+m}; j = 0, 1, 2, \dots\} \subset \Lambda_m^*.$$

On the other hand, both C_{φ_a} and $C_{\varphi_a}^*$ are invertible on $H^2(D)$, then

$$\overline{\text{span}}\{C_{\varphi_a} z^k; k = 0, 1, 2, \dots\} = \overline{\text{span}}\{C_{\varphi_a}^* z^k; k = 0, 1, 2, \dots\} = H^2(D).$$

so we have

$$\Lambda_m = \overline{\text{span}}\{C_{\varphi_a} z^{3j+m}; j = 0, 1, 2, \dots\}$$

and

$$\Lambda_m^* = \overline{\text{span}}\{C_{\varphi_a}^* z^{3j+m}; j = 0, 1, 2, \dots\}.$$

Finally, $C_{\varphi_a} z^k = \varphi_a^k$ and the proof of Lemma 2.2 in [2] shows that $C_{\varphi_a} z^k = e_k - a e_{k-1}$, hence the proof is done. \square

Remark 2.5. Note that $\|e_j\|^2 = (1 - |a|^2)^{-1}$ for $j = 0, 1, 2, \dots$ and $\langle e_j, e_k \rangle = 0$ whenever $j \neq k$.

Remark 2.6. If C_{φ} is C -symmetric with respect to a conjugation C , then $C\Lambda_m = \Lambda_m^*$ for $m = 0, 1, 2$.

3. PROOF OF THE MAIN RESULT

In this section, we will focus on the proof of our main result, i.e., Theorem 3.5, which assert that no elliptic automorphism of order 3 except for rotations can induce a complex symmetric composition operator on $H^2(D)$.

We would like to point out here that throughout the rest of this paper, each notation will always represent the same thing as it did initially. For example, φ is always a elliptic automorphism of order 3 in what follows, a is always the fixed point of φ in $D \setminus \{0\}$, and ρ always represents the same constant $-\frac{\bar{a}^2}{a} \cdot \frac{1-|a|^2}{1-|a|^4}$.

We will assume that C_{φ} is C -symmetric with respect to some conjugation C , and finally we will show this assumption leads to a contradiction. Now we start by determining the image of a certain vector under the conjugation C . The notation $\{e_j\}_{j=0}^{\infty}$ in Lemma 2.4 is still valid in this section.

Claim 3.1. *Let φ be an elliptic automorphism of order 3 with fixed point $a \in D \setminus \{0\}$. If C_{φ} is C -symmetric on $H^2(D)$ with respect to a conjugation C , then we have*

$$C e_0 = c_0 \frac{1 - \bar{a}^3 \varphi_a^3}{1 - \rho \varphi_a^3},$$

where c_0 is a constant and $\rho = -\frac{\bar{a}^2}{a} \cdot \frac{1-|a|^2}{1-|a|^4}$.

Proof. Let $h_0 = C e_0$. Since $e_0 \in \Lambda_0^*$, we have $h_0 \in \Lambda_0$. So we can suppose that

$$h_0 = \sum_{j=0}^{\infty} c_j \varphi_a^{3j}.$$

It is obvious that e_0 is orthogonal to Λ_2^* . So by Remark 2.6 and the fact that C is an isometry, h_0 is orthogonal to Λ_2 , which means that $\langle h_0, \varphi_a^{3k+2} \rangle = 0$ for $k = 0, 1, 2, \dots$. So we have the following equations,

$$(3.1) \quad \sum_{j=0}^k c_j \bar{a}^{3k+2-3j} + \sum_{j=k+1}^{\infty} c_j a^{3j-3k-2} = 0$$

for $k = 0, 1, 2, \dots$. Replace k by $k + 1$ in (3.1) we get

$$\sum_{j=0}^{k+1} c_j \bar{a}^{3k+5-3j} + \sum_{j=k+2}^{\infty} c_j a^{3j-3k-5} = 0,$$

hence

$$(3.2) \quad \sum_{j=0}^{k+1} c_j \bar{a}^{3k+5-3j} a^3 + \sum_{j=k+2}^{\infty} c_j a^{3j-3k-2} = 0.$$

Combining (3.1) and (3.2), we have

$$(3.3) \quad \sum_{j=0}^k c_j \bar{a}^{3k+2-3j} + c_{k+1} a \frac{1 - |a|^4}{1 - |a|^6} = 0$$

for $k = 0, 1, 2, \dots$. Thus, we have $c_1 = \tilde{\rho}c_0$ and $c_{j+1} = \rho c_j$ for $j = 1, 2, 3, \dots$, where $\tilde{\rho} = -\frac{\bar{a}^2}{a} \cdot \frac{1-|a|^6}{1-|a|^4}$ and $\rho = -\frac{\bar{a}^2}{a} \cdot \frac{1-|a|^2}{1-|a|^4}$. Therefore,

$$\begin{aligned} h_0 &= c_0 + c_1 \sum_{j=1}^{\infty} \rho^{j-1} \varphi_a^{3j} \\ &= c_0 + c_0 \frac{\tilde{\rho} \varphi_a^3}{1 - \rho \varphi_a^3} \\ &= c_0 \frac{1 - \bar{a}^3 \varphi_a^3}{1 - \rho \varphi_a^3}. \end{aligned}$$

□

Claim 3.2. For the constant c_0 in Claim 3.1, we have

$$|c_0| = \frac{1}{1 - |a|^4}.$$

Proof. Let

$$g = \frac{\bar{p} - \varphi_a^3}{1 - \rho \varphi_a^3},$$

then g is an inner function and $g(0) = -\frac{a^2}{\bar{a}}$.

A easy calculation shows that $h_0 = \gamma_1 g + \gamma_2$, where

$$\gamma_1 = c_0 \frac{\bar{a}^2}{a} (1 + |a|^2), \gamma_2 = c_0 (1 + |a|^2).$$

So we have

$$\begin{aligned} \|h_0\|^2 &= \langle \gamma_1 g + \gamma_2, \gamma_1 g + \gamma_2 \rangle \\ &= |\gamma_1|^2 + |\gamma_2|^2 + 2\Re\{\gamma_1 \overline{\gamma_2} g(0)\} \\ &= |c_0|^2 (1 - |a|^2)(1 + |a|^2)^2. \end{aligned}$$

Since C is isometric, we can know that

$$\|h_0\|^2 = \|e_0\|^2 = \frac{1}{1 - |a|^2},$$

thus $|c_0| = (1 - |a|^4)^{-1}$.

□

Claim 3.3. For the function $h_0 = Ce_0$ in the proof of Claim 3.1, we have

$$\langle h_0, \varphi^{3k} \rangle = c_0(1 - |a|^4)\rho^k$$

for $k = 0, 1, 2, \dots$

Proof.

$$(3.4) \quad \langle h_0, \varphi^{3k} \rangle = \sum_{j=0}^k c_j \bar{a}^{3k-3j} + \sum_{j=k+1}^{\infty} c_j a^{3j-3k}.$$

Comparing (3.4) with (3.1), we can get that

$$(3.5) \quad \langle h_0, \varphi^{3k} \rangle = (1 - |a|^4) \sum_{j=0}^k c_j \bar{a}^{3k-3j}.$$

So (3.5), along with (3.3), shows that

$$\begin{aligned} \langle h_0, \varphi^{3k} \rangle &= (1 - |a|^4) \frac{c_{k+1}}{\tilde{\rho}} \\ &= c_0(1 - |a|^4)\rho^k. \end{aligned}$$

□

Claim 3.4. Under the assumption of Claim 3.1, we have

$$Ce_1 = -c_0 \frac{\bar{a}(1 - |a|^6)}{a(1 - |a|^4)} \cdot \frac{\varphi_a(1 - \bar{a}^3 \varphi_a^3)}{(1 - \rho \varphi_a^3)^2} + \bar{a}h_0.$$

Proof. Let $h_1 = Ce_1$. Since $e_1 - ae_0 \in \Lambda_1^*$, we have $h_1 - \bar{a}h_0 \in \Lambda_1$. So we can assume that

$$h_1 = \sum_{j=0}^{\infty} b_j \varphi_a^{3j+1} + \bar{a}h_0.$$

It is obvious that e_1 is orthogonal to Λ_0^* , so h_1 is orthogonal to Λ_0 , which means that $\langle h_1, \varphi_a^{3k} \rangle = 0$ for $k = 0, 1, 2, \dots$ So we have the following equations,

$$\sum_{j=0}^{\infty} b_j a^{3j+1} + c_0 \bar{a}(1 - |a|^4) = 0,$$

and

$$\sum_{j=0}^{k-1} b_j \bar{a}^{3k-3j-1} + \sum_{j=k}^{\infty} b_j a^{3j+1-3k} + c_0 \bar{a}(1 - |a|^4)\rho^k = 0$$

for $k = 1, 2, 3, \dots$

So

$$b_0 a \frac{1 - |a|^4}{1 - |a|^6} + c_0 \bar{a} = 0,$$

and

$$\sum_{j=0}^{k-1} b_j \bar{a}^{3k-3j-1} + b_k a \frac{1 - |a|^4}{1 - |a|^6} + c_0 \bar{a} \rho^k = 0$$

for $k = 1, 2, 3, \dots$

Now let

$$\delta_j = -\frac{b_j a(1 - |a|^4)}{c_0 \bar{a}(1 - |a|^6)},$$

then $\delta_0 = 1$, $\delta_1 = \rho + \tilde{\rho}$, and

$$\delta_{k+1} = \rho\delta_k + \tilde{\rho}\rho^k$$

for $k = 1, 2, 3, \dots$. Hence we can get that

$$\delta_k = \rho^k + k\tilde{\rho}\rho^{k-1}$$

for $k = 0, 1, 2, \dots$

Thus we have

$$\begin{aligned} h_1 - \bar{a}h_0 &= \sum_{j=0}^{\infty} b_j \varphi_a^{3j+1} \\ &= -c_0 \frac{\bar{a}(1 - |a|^6)}{a(1 - |a|^4)} \sum_{j=0}^{\infty} \delta_j \varphi_a^{3j+1} \\ &= -c_0 \frac{\bar{a}(1 - |a|^6)}{a(1 - |a|^4)} \left(\sum_{j=0}^{\infty} \rho^j \varphi_a^{3j+1} + \sum_{j=0}^{\infty} j\tilde{\rho}\rho^{j-1} \varphi_a^{3j+1} \right) \\ &= -c_0 \frac{\bar{a}(1 - |a|^6)}{a(1 - |a|^4)} \left(\frac{\varphi_a}{1 - \rho\varphi_a^3} + \frac{\tilde{\rho}\varphi_a^4}{(1 - \rho\varphi_a^3)^2} \right) \\ &= -c_0 \frac{\bar{a}(1 - |a|^6)}{a(1 - |a|^4)} \cdot \frac{\varphi_a(1 - \bar{a}^3\varphi_a^3)}{(1 - \rho\varphi_a^3)^2}. \end{aligned}$$

□

Now we can prove our final result as follows.

Theorem 3.5. *If φ is an elliptic automorphism of order 3 with fixed point $a \in D \setminus \{0\}$, then C_φ is not complex symmetric on $H^2(D)$.*

Proof. Suppose that C_φ is C -symmetric with respect to conjugation C . Then Claim 3.1 to 3.4 hold.

Let

$$f = \frac{1 - |a|^6}{1 - |a|^4} \cdot \frac{1 - \bar{a}^3\varphi_a^3}{(1 - \rho\varphi_a^3)^2},$$

Then $\|f\| = |c_0|^{-1} \cdot \|h_1 - \bar{a}h_0\| = (1 - |a|^4)\|h_1 - \bar{a}h_0\|$.

A tedious calculation shows that $f = \beta_1 g^2 + \beta_2 g + \beta_3$, where $g = \frac{\bar{\rho} - \varphi_a^3}{1 - \rho\varphi_a^3}$ and

$$\begin{aligned} \beta_1 &= \frac{(1 - |a|^4)(1 + |a|^2)^2}{1 - |a|^6} (\rho^2 - \bar{a}^3\rho) = \frac{\bar{a}^4}{a^2} (1 + |a|^2); \\ \beta_2 &= \frac{(1 - |a|^4)(1 + |a|^2)^2}{1 - |a|^6} (-2\rho + \bar{a}^3 + \bar{a}^3|\rho|^2) = \frac{\bar{a}^2}{a} (1 + |a|^2)(2 + |a|^2); \\ \beta_3 &= \frac{(1 - |a|^4)(1 + |a|^2)^2}{1 - |a|^6} (1 - \bar{a}^3\bar{\rho}) = (1 + |a|^2)^2. \end{aligned}$$

So

$$\begin{aligned} \|f\|^2 &= \langle \beta_1 g^2 + \beta_2 g + \beta_3, \beta_1 g^2 + \beta_2 g + \beta_3 \rangle \\ &= |\beta_1|^2 + |\beta_2|^2 + |\beta_3|^2 + 2\Re(\beta_1 \bar{\beta}_2 g(0) + \beta_2 \bar{\beta}_3 g(0) + \beta_1 \bar{\beta}_3 g(0)^2) \\ &= (1 + 2|a|^2 - 2|a|^4 - |a|^6)(1 + |a|^2)^2. \end{aligned}$$

However, since C is isometric,

$$\begin{aligned} \|f\|^2 &= (1 - |a|^4)^2 \|h_1 - \bar{a}h_0\|^2 \\ &= (1 - |a|^4)^2 \|e_1 - \bar{a}e_0\|^2 \\ &= (1 - |a|^4)^2 (1 + |a|^2)(1 - |a|^2) \\ &= (1 - |a|^4)(1 + |a|^2)^2. \end{aligned}$$

So we have

$$\begin{aligned} (1 + 2|a|^2 - 2|a|^4 - |a|^6)(1 + |a|^2)^2 &= (1 - |a|^4)(1 + |a|^2)^2 \\ 2|a|^2 - |a|^4 - |a|^6 &= 0 \\ |a|^2 + |a|^4 &= 2, \end{aligned}$$

which is impossible since $a \in D$. \square

At last we can get our main result as a corollary. It gives a complete description of the automorphisms which can induce complex symmetric operators on $H^2(D)$.

Corollary 3.6. *Suppose φ is an automorphism of D . Then C_φ is complex symmetric on $H^2(D)$ if and only if φ is either a rotation or an elliptic automorphism of order two.*

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